# Triangles with ideal proportions of sides

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**Abstract.** We discuss the properties of triangles whose sides establish a geometric sequence, paying special attention to their connections with the golden ratio and issues related to the Fibonacci numbers. We also pose a number of questions concerning the unions of triangles of the considered type.

**Keywords:** ideal proportions, golden ratio, Kepler triangle, Fibonacci numbers, Lucas numbers.

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## 1. Proportions and angles

The idea of the golden ratio led us to discussion on a class of triangles we called *triangles with ideal proportions of sides* or, in brief, *ideal triangles*. Consider a triangle with sides $a \le b \le c$ and angles $\alpha$, $\beta$ and $\gamma$, opposite to the sides $a$, $b$ and $c$, respectively. Since the longer side is across from the larger angle, $\alpha \le \beta \le \gamma$.

We say that a triangle with sides $a \le b \le c$ has an ideal proportion of sides if the following equality holds

$$\frac{c}{b} = \frac{b}{a}. \tag{1}$$

Denote the ratio referred to in (1) by $y$. Then

$$b = ay, \quad c = ay^2. \tag{2}$$

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While creating this paper I. Sobstyl was a student of the Technical High School of Computer Science. He worked under scientific supervision (in the form of individual course in Mathematics) of Professor R. Wituła.

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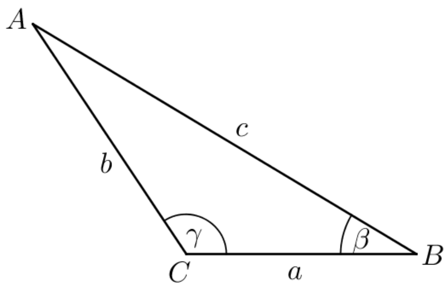


Fig. 1. Triangle under discussion

As a consequence of (2) and the triangle inequality we obtain the following estimation

$$1 \le y < \phi, \tag{3}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

In the limit case, i.e. for $y = \phi$, we get $c = \phi^2 a = (\phi + 1)a$, $b = \phi a$. So, the limit analogue of a triangle is the interval of length $c = (\phi + 1)a$ divided into two subintervals of lengths $b = \phi a$ and $a$.

Applying the law of cosines to the angles of an ideal triangle with ratio $y$ we get[1]

$$y^4 - 2y^3 \cos\alpha + y^2 - 1 = 0, \tag{4}$$

$$y^4 - (2\cos\beta + 1)y^2 + 1 = 0, \tag{5}$$

$$y^4 - y^2 + 2y\cos\gamma - 1 = 0. \tag{6}$$

Clearly $\alpha \in \left(0, \frac{\pi}{3}\right]$ and $\gamma \in \left[\frac{\pi}{3}, \pi\right)$.

The existance of a solution $y$ of equation (5) yields

$$\cos\beta \ge \frac{1}{2}. \tag{7}$$

**Proposition 1.1.** *In an ideal triangle, the medium angle $\beta$ has measure at most $\frac{\pi}{3}$.*

**Example 1.2.** The triangle with angles 95°, 70° and 15° is not an ideal triangle.

[1] It follows directly from equations (4)-(6) that

$$2(y\cos\gamma + y^2\cos\beta + y^4\cos\alpha) = 1 + y^2 + y^4$$

which is a particular form of the identity $2(ab\cos\gamma + ac\cos\beta + bc\cos\alpha) = a^2 + b^2 + c^2$ holding for all triangles. Nevertheless, ideal triangles satisfy besides the following equalities:

$$a\cos\gamma + b\cos\beta + c\cos\alpha = \frac{a^2+b^2+c^2}{2b},$$

$$\frac{\cos\alpha}{a} + \frac{\cos\beta}{b} + \frac{\cos\gamma}{c} = abc\left(\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4}\right).$$

Recall that $y \geq 1$. Therefore the only solution of equation (5) is

$$y = \sqrt{\cos\beta + \frac{1}{2} + \sqrt{\left(\cos\beta + \frac{1}{2}\right)^2 - 1}}. \tag{8}$$

This means that if $\beta$ is given then one can compute the measures of the other angles.

One can easily check that for fixed $\alpha \in \left(0, \frac{\pi}{3}\right]$ equation (4) has exactly one solution $y \in [1, \phi)$. Indeed, the function $g(y) = y^4 - 2y^3 \cos\alpha + y^2 - 1$ is increasing in $(0, \infty)$, $g(1) = 1 - 2\cos\alpha \leq 0$ and $g(\phi) = (4\phi + 2)(1 - \cos\alpha) > 0$.

Combining the above observations we obtain the following statement.

**Proposition 1.3.** *Assume that $\theta \in \left(0, \frac{\pi}{3}\right]$. Then*

- *there is exactly one ideal triangle with the smallest angle $\alpha = \theta$ and the shortest side $a = 1$,*
- *there is exactly one ideal triangle with the medium angle $\beta = \theta$ and the shortest side $a = 1$.*

In like manner the measure of the largest angle determines the proportion of an ideal triangle.

Equation (6) can be rewritten in the form

$$2\cos\gamma = \frac{1 + y^2 - y^4}{y}. \tag{9}$$

Define the function

$$f(x) := \frac{1 + x^2 - x^4}{x}, \quad x \neq 0, \tag{10}$$

that will be discussed further in Section 4. Note that $f$ is decreasing in each of the intervals $(-\infty, 0)$ and $(0, \infty)$, injective, $f(1) = 1 = 2\cos\frac{\pi}{3}$ and $f(\phi) = -2 = 2\cos\pi$, so $y(\gamma) = f^{-1}(2\cos\gamma)$ is well defined in $[\frac{\pi}{3}, \pi]$ and increasing as the composition of two decreasing functions.

**Proposition 1.4.** *For every $\gamma \in [\frac{\pi}{3}, \pi)$ there is exactly one – up to similarity – ideal triangle whose largest angle is of measure $\gamma$. Moreover, the function $y(\gamma)$ is strictly increasing.*

**Corollary 1.5.** *The Kepler triangle (see [2]) with ratio $y = \sqrt{\phi}$ is the only right triangle with an ideal proportion of sides.*

**Corollary 1.6.** *Every ideal triangle having an angle of measure $\frac{\pi}{3}$ is equilateral.*

## 2. Pairs of conjugate ratios

Two ideal triangles are said to be *conjugated* if the sum of their largest angles equals $\pi$. A pair of numbers $(y, y^*)$ is called a *pair of conjugated ratios* if there are conjugated ideal triangles with ratios $y$ and $y^*$, respectively. Denote by $\gamma$ the largest angle of the triangle with ratio $y$. Then, according to (9) and (10),

$$f(y) = 2\cos\gamma, \tag{11}$$
$$f(y^*) = 2\cos(\pi-\gamma) = -2\cos\gamma. \tag{12}$$

Directly from the definition of conjugated triangles we conclude that their largest angles cannot exceed $\frac{2\pi}{3}$. On the other hand, by Propostion 1.4 equation (12) has a unique solution $y^*$ for every $\gamma \in [\frac{\pi}{3}, \frac{2\pi}{3}]$. Thus there is exactly one conjugate ratio $y^*$ for every $y \in [1,\psi]$, where $\psi$ is the ratio $y$ associated with the angle $\frac{2\pi}{3}$.

**Example 2.1.** Observe that $\psi = y^*$ for $\gamma = \frac{\pi}{3}$ and $y = 1$. Then we get

$$(y^*)^4 - (y^*)^2 - y^* - 1 = 0$$

i.e.

$$(y^*)^3 - (y^*)^2 - 1 = 0,$$
$$1 - \frac{1}{y^*} - \frac{1}{(y^*)^3} = 0.$$

From the Cardano formula (see [5], [9]) we get

$$\frac{1}{y^*} = \sqrt[3]{\frac{1+\sqrt{\frac{31}{27}}}{2}} + \sqrt[3]{\frac{1-\sqrt{\frac{31}{27}}}{2}} = \sqrt[3]{\frac{1+\sqrt{\frac{31}{27}}}{2}} - \sqrt[3]{\frac{\sqrt{\frac{31}{27}}-1}{2}}$$

which implies $\psi = y^* \approx 1.46557124$. We note that $\frac{1}{y^*} \approx 0.682328$ is the fixed point of the following scaled Perrin polynomial $p(x) := -\frac{1}{2}(x^3 - x - 1)$ (see [8]).

It is easily seen that the function assigning $y^*$ to $y$ is an involution. Its fixed point is $\sqrt{\phi}$, the ratio of the Kepler triangle, the only self-conjugated ideal triangle.

**Proposition 2.2.** *If $y^*$ is the ratio conjugated with $y \in [1,\psi]$, then*

$$\frac{1}{yy^*} + 1 - y^2 - (y^*)^2 + yy^* = 0. \tag{13}$$

*Proof.* If $(y, y^*)$ is a pair of conjugated ratios then

$$f(y) + f(y^*) = 0,$$

$$\frac{1}{y} + y + y^3 + \frac{1}{y^*} + y^* + (y^*)^3 = 0,$$

$$\frac{y+y^*}{yy^*} + (y+y^*) - (y+y^*)(y^2 - yy^* + (y^*)^2) = 0,$$

which implies (13). □

Describing $(y, y^*)$ in (13) in the polar coordinates

$$y = \lambda(t)\cos t, \quad y^* = \lambda(t)\sin t$$

we get

$$\frac{2}{\lambda^2 \sin 2t} + 1 - \left(1 - \frac{1}{2}\sin 2t\right)\lambda^2 = 0 \tag{14}$$

which implies

$$\lambda = \sqrt{\frac{\sin 2t + \sqrt{8\sin 2t - 3\sin^2 2t}}{(\sin 2t)(2 - \sin 2t)}}, \tag{15}$$

where $t \in \left[\arcsin \frac{1}{\sqrt{1+\psi^2}}, \arcsin \frac{\psi}{\sqrt{1+\psi^2}}\right]$. The latter equality can be used to draw the graph of the function $y^*(y)$.

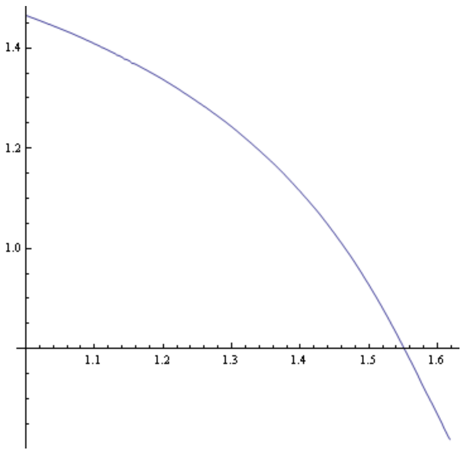


Fig. 2. Graph of $y^*(y)$ function

## 3. Ideal triangles and the golden ratio

The basic connection between the golden ratio and the proportions of ideal triangles is given in (3). In the following we discuss several other properties of ideal triangles which are associated to the golden ratio.

One of the nicest properties of the Kepler triangle is the fact that its altitude dropped of the right-angled vertex divides its hypotenuse in the golden ratio, as it can be seen in Figure 3. It turns out that no other triangle with ideal proportions of sides has this property.

**Proposition 3.1.** *The Kepler triangle is the only ideal triangle whose longest side is divided by the altitude in golden ratio.*

*Moreover, in any ideal triangle the altitude dropped onto its longest side $c$ divides this side into segments $c_1$ and $c_2$, $c_1 \geq c_2$, in such a way that*

$$\frac{c}{c_1} - \phi \in \left[\frac{8}{5} - \phi, 2 - \phi\right] \approx [-0.01803, 0.3819].$$

*Proof.* Consider an ideal triangle with ratio $y = \frac{c}{b} = \frac{b}{a}$. Denote by $h$ its altitude dropped on side $c$ (see Figure 4). Using the Pythagorean Theorem we get the following system of equations

$$\begin{cases} h^2 + c_1^2 = b^2, \\ h^2 + (c - c_1)^2 = a^2. \end{cases}$$

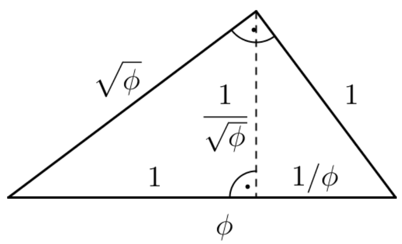


Fig. 3. The Kepler triangle

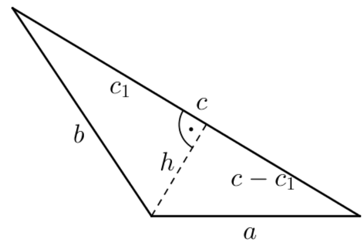


Fig. 4. Triangle under discussion

Subtracting the first equations from the second one we obtain the equality

$$c^2 - 2c_1c = a^2 - b^2.$$

Since $b = ac$, we get

$$\frac{c}{c_1} = \frac{2c^2}{c^2 + ac - a^2} = \frac{2}{1 + \frac{a}{c} - \left(\frac{a}{c}\right)^2}. \tag{16}$$

Note that

$$\frac{c}{c_1} - \phi = \phi\frac{\frac{2}{\phi} - 1 - \frac{a}{c} + \left(\frac{a}{c}\right)^2}{1 + \frac{a}{c} - \left(\frac{a}{c}\right)^2} = \phi\frac{(\frac{a}{c} - \frac{1}{\phi})(\frac{a}{c} - \frac{1}{\phi^2})}{1 + \frac{a}{c} - \left(\frac{a}{c}\right)^2}. \tag{17}$$

Therefore $\frac{c}{c_1} = \phi$ only if $\frac{a}{c} - \frac{1}{\phi} = 0$, i.e. when $y = \sqrt{\phi}$ and the considered triangle is the Kepler triangle. In case $\frac{c}{a} = \phi^2$ we obtain $y = \phi$ which contradicts (3).

Recall that $\frac{a}{c} = \frac{1}{y^2}$ and $y \in [1, \phi)$. Then $\frac{a}{c} \in \left(\frac{1}{\phi^2}, 1\right]$, which implies that the polynomial $1 + \frac{a}{c} - \left(\frac{a}{c}\right)^2$, the denominator of (16), attains its maximum $\frac{5}{4}$ at $\frac{a}{c} = \frac{1}{2}$ and its minimum 1 at $\frac{a}{c} = 1$. Therefore $\left[\frac{8}{5}, 2\right]$ is the set of all possible values of $\frac{c}{c_1}$ and for every ideal triangle

$$\frac{8}{5} - \phi \leq \frac{c}{c_1} - \phi \leq 2 - \phi.$$

□

Obviously, the longest side $c$ can divided in golden ratio by one of two possible points. In the following we calculate the lengths of segments joining such points to the opposite vertex.

**Proposition 3.2.** *Assume that $D_1$ divides the longest side $AB$ of an ideal triangle $ABC$ in such a way that $\frac{|AB|}{|AD_1|} = \frac{|AD_1|}{|BD_1|}$ (see Figure 5). Then*

$$|CD_1| = \frac{a}{\phi}\sqrt{\left(1+\frac{c}{a}\right)\left(\phi - \frac{1}{\phi}\cdot\frac{c}{a}\right)}.$$

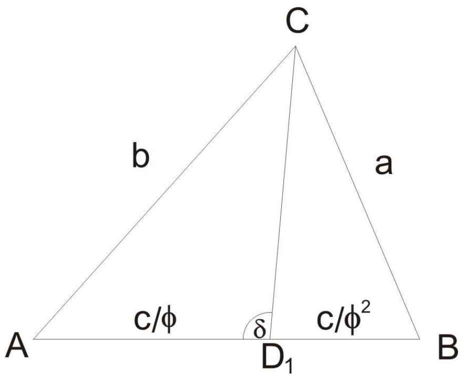


Fig. 5. The point $D_1$ discussed in Proposition 3.2

*Proof.* Denote the length of $CD_1$ by $d$. From the Law of Cosines we get

$$\begin{cases} a^2 = d^2 + \frac{1}{\phi^4}c^2 - 2\frac{dc}{\phi^2}\cos\delta, \\ \frac{1}{\phi}b^2 = \frac{1}{\phi}d^2 + \frac{1}{\phi^3}c^2 + 2\frac{dc}{\phi^2}\cos\delta, \end{cases}$$

from which, by adding both equations, we obtain

$$a^2 + \frac{1}{\phi}b^2 = \left(1+\frac{1}{\phi}\right)d^2 + \frac{1}{\phi^3}\left(1+\frac{1}{\phi}\right)c^2 = \phi d^2 + \frac{1}{\phi^2}c^2.$$

Since $ac = b^2$ we deduce

$$\begin{aligned} d^2 &= \frac{1}{\phi}\left(a^2 + \frac{1}{\phi}ac - \frac{1}{\phi^2}c^2\right) = \frac{a^2}{\phi^3}\left(\phi^2 + \phi\frac{c}{a} - \left(\frac{c}{a}\right)^2\right) \\ &= \frac{a^2}{\phi^2}\left(1+\frac{c}{a}1\right)\left(\phi - \frac{1}{\phi}\cdot\frac{c}{a}\right) \end{aligned}$$

which ends the proof. $\square$

**Proposition 3.3.** *Assume that $D_2$ divides the longest side $AB$ of an ideal triangle $ABC$ in such a way that $\frac{|AB|}{|BD_2|} = \frac{|BD_2|}{|AD_2|}$ (see Figure 6). Then*

$$|CD_2| = \frac{a}{\phi}\sqrt{1 + \phi \cdot \frac{c}{a} - \frac{1}{\phi} \cdot \left(\frac{c}{a}\right)^2}.$$

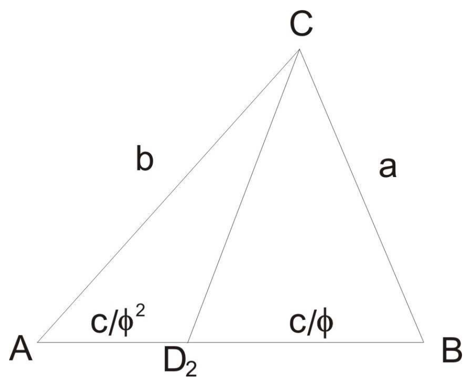


Fig. 6. The point $D_2$ considered in Proposition 3.3

*Proof.* The proof is similar to the one of Proposition 3.2. □

**Corollary 3.4.** *If $D_1$ and $D_2$ are two different point dividing the longest side $c$ of an ideal triangle $ABC$ in the golden ratio then*

$$\big||CD_1|^2 - |CD_2|^2\big| = \frac{1}{\phi^3}\left(b^2 - a^2\right), \tag{18}$$

$$|CD_1|^2 + |CD_2|^2 = a^2 + b^2 - \frac{2}{\phi^3} \cdot c^2. \tag{19}$$

*Directly from* (19) *we get*

$$|CD_1|^2 + |CD_2|^2 = a^2 + b^2 - 2|AD_1| \cdot |BD_1| = a^2 + b^2 - 2|AD_2| \cdot |BD_2|.$$

## 4. Some properties of function $f(x)$

In this section we discuss three analytic properties of function

$$f(x) = \frac{1 + x^2 - x^4}{x}, \quad x \neq 0,$$

defined in Section 1.

**Observation 4.1.** *Function $f(x)$ is odd, convex in $\left(-\sqrt[4]{\frac{1}{3}}, 0\right) \cup \left(\sqrt[4]{\frac{1}{3}}, \infty\right)$ and concave in $\left(-\infty, -\sqrt[4]{\frac{1}{3}}\right) \cup \left(0, \sqrt[4]{\frac{1}{3}}\right)$. It has zeros at points $\sqrt{\phi}$ and $-\sqrt{\phi}$.*

*Proof.* Clearly $f(-x) = -f(x)$ and $f(\sqrt{\phi}) = 0$. Since $f$ is decreasing in $(-\infty, 0)$ and $(0, \infty)$, it has no zeros except $\pm\sqrt{\phi}$.

It is easy to verify that $f''(x) = \frac{2}{x^3}(1 - 3x^4) < 0$ if and only if $x \in \left(-\sqrt[4]{\frac{1}{3}}, 0\right) \cup \left(\sqrt[4]{\frac{1}{3}}, \infty\right)$. □

**Remark 4.2.** In the following we use general relations linking the powers of $\phi$ and Fibbonacci numbers (for more information see [3, 6, 10, 7]). Namely,

$$\phi^n = F_n\phi + F_{n-1}, \quad n \in \mathbb{Z}, \tag{20}$$

where $F_n$ denotes the $n$-th Fibonacci number. Recall that $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for every $n \in \mathbb{Z}$. Directly from this definition we obtain $F_{-n} = (-1)^{n+1}F_n$ for $n \in \mathbb{N}$. Hence (20) for negative $n$ follows from the formula for positive numbers and Cassini's identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$, $n \in \mathbb{N}$.[2]

**Observation 4.3.** *The equation*

$$f(x+t) - f(x) = f(x) - f(x-t) \tag{21}$$

*has a positive solution $t(x)$ if and only if $x \in \left(-\infty, -\sqrt[4]{\frac{1}{3}}\right) \cup \left(\sqrt[4]{\frac{1}{3}}, \infty\right)$. In this case $t(x) = \sqrt{x^2 - \frac{1}{3} \cdot x^{-2}}$.*

*For instance,*

- $t(\sqrt{\phi}) = \sqrt{\frac{1}{3}(2\phi + 1)} = \sqrt{\frac{2+\sqrt{5}}{3}}$,
- $t(\phi^{k/2}) = \sqrt{\frac{1}{3}\left[(3 + (-1)^k)F_k\phi + (3 - (-1)^k)F_{k-1} + (-1)^{k+1}F_k\right]}$
$= \begin{cases} \sqrt{\frac{1}{3}\left[(3 + (-1)^k)F_k\phi + (-1)^{k+1}F_{k-1} + F_{k-3}\right]} & \text{if } k \text{ is even} \\ \sqrt{\frac{1}{3}\left[(3 + (-1)^k)F_k\phi + (-1)^{k+1}F_{k-1} + L_{k+1} - F_{\frac{k-1}{2}}\right]} & \text{if } k \text{ is odd} \end{cases}$,
*where $L_k$ denotes the Lucas numbers ($L_0 = 2$, $L_1 = 1$, $L_{n+1} = L_n + L_{n-1}$, for every $n \in \mathbb{Z}$),*
- $t\left(\sqrt[4]{\frac{1}{3}} \cdot \phi^{k/2}\right) = \sqrt[4]{3}\sqrt{(1 + (-1)^k)F_k\phi + F_k + (1 + (-1)^{k+1})F_{k-1}}$.
*In the last case,*
$t\left(\sqrt[4]{\frac{1}{3}} \cdot \phi^{k/2}\right) = \sqrt[4]{3}\sqrt{F_k + 2F_{k-1}} = \sqrt[4]{3}\sqrt{L_k}$ *if $k$ is odd and*
$t\left(\sqrt[4]{\frac{1}{3}} \cdot \phi^{k/2}\right) = \sqrt[4]{3}\sqrt{F_k(2\phi + 1)} = \sqrt[4]{3} \cdot \phi\sqrt{\phi F_k}$ *if $k$ is even.*

Rather technical but straightforward proof of the above assertions is omitted here.

Another interesting property concerns the differences of the areas between the graph of $f$ and the $x$-axis for adjacent segments $[a, b]$ and $[b, c]$ with $a$, $b$, $c$ satisfying (1).

[2] The observation concerning $\phi^n$ and Fibonacci numbers for negative $n$ is probably original.

**Observation 4.4.** *Assume that* $0 < a < b < c$ *and put*

$$P_1 = \int_a^b f(x)dx \quad \text{and} \quad P_2 = \int_b^c f(x)dx.$$

*If* $\frac{b}{a} = \frac{c}{b}$ *then*

$$P_1 - P_2 = \frac{1}{4}(c-a)^2\left((c+a)^2 - 2\right). \tag{22}$$

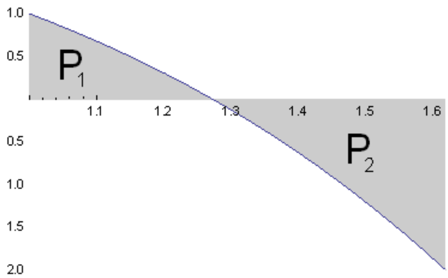


Fig. 7. $P_1$ and $P_2$ for $a = 1$, $b = \sqrt{\phi}$, $c = \phi$

*In particular,*

- *if* $a = b \cdot \phi^{-1/2}$, $c = b \cdot \phi^{1/2}$ *then*

$$P_1 - P_2 = \frac{1}{4}b^2(b^2 - 2\phi^{-3}) = \frac{1}{4}b^2(b^2 - 4\phi + 6) \tag{23}$$

- *if* $a = 1$, $b = \sqrt{\phi}$, $c = \phi$ *then*

$$P_1 - P_2 = \frac{3}{4}\phi^{-1} \tag{24}$$

- *if* $a = b \cdot \phi^{-1}$, $c = b \cdot \phi$ *then*

$$P_1 - P_2 = \frac{1}{4}b^2(5b^2 - 2) \tag{25}$$

- *if* $k \in \mathbb{N}$, $a = \phi^{-k/2}$, $b = 1$, $c = \phi^{k/2}$ *then*

$$P_1 - P_2 = \frac{1}{4}(\phi^k - 1)^2 + \frac{1}{4}(\phi^{-k} - 1)^2 = \begin{cases} \frac{1}{4}L_{2k} - \frac{1}{2}L_k + \frac{1}{2} & \text{if } k \text{ is even} \\ \frac{1}{4}L_{2k} - \frac{\sqrt{5}}{2}F_k + \frac{1}{2} & \text{if } k \text{ is odd} \end{cases}. \tag{26}$$

*On the other hand, using* (20) *we get*

$$P_1 - P_2 = \frac{1}{4}(2 + F_{2k+1} + F_{2k-1}) - \frac{1}{2}(1 + (-1)^{k+1})F_k\phi - \frac{1}{2}(1 + (-1)^k)F_{k-1} - \frac{1}{2}(-1)^k F_k.$$

*If $k$ is odd then*

$$P_1 - P_2 = \frac{1}{4}(2 + F_{2k+1} + F_{2k-1}) - F_k\phi + \frac{1}{2}F_k.$$

*If $k$ is even then*

$$P_1 - P_2 = \frac{1}{4}(2 + F_{2k+1} + F_{2k-1}) - F_{k-1} - \frac{1}{2} \cdot F_k.$$

## 5. Dividing triangles into ideal triangles

While working on the paper we asked several questions which became a challenge to us and finally we have not solved them.

**Problem 1.** *Describe all triangles which can be divided into two ideal triangles.*

It turns out that not all triangles admit this property. If we cut equilateral triangle into two triangles, then each of them will have an angle of measure $\frac{\pi}{3}$. Since the smaller triangles are not equilateral, they are not ideal (see Corollary 1.6).

On the other hand, the altitude of the Kepler triangle divides it into two triangles which are congruent to the initial one and thus ideal (see Figure 3).

Gluing two congruent Kepler triangles along their catheti of the same length results in two isosceles triangles which are clearly the unions of two conjugated ideal triangles. Both of them are depicted in Figure 8.

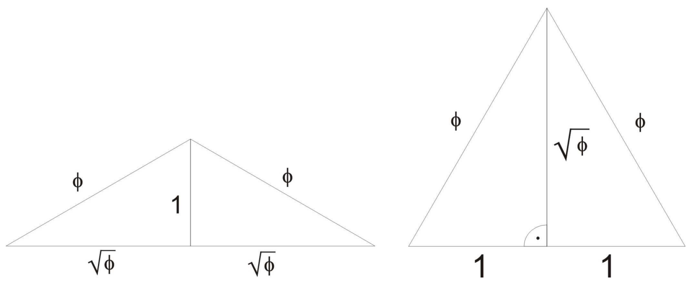


Fig. 8. Unions of two Kepler triangles

These examples led us to the question concerning isosceles triangles.

**Problem 2.** *Characterize all isosceles triangles which can be cut into two ideal triangles.*

Now we come back to the concept of conjugacy of triangles (Section 2). Actually, this notion came into existence in order to simplify the problem of splitting triangles into two adjacent ideal triangles.

**Problem 3.** *Describe all triangles which can be divided into two conjugated ideal triangles.*

Another question concerns the possibility of cutting a triangle into three ideal triangles.

**Problem 4.** *Which triangles $ABC$ contain a point $P$ such that all triangles $ABP$, $ACP$ and $BCP$ have ideal proportions of sides?*

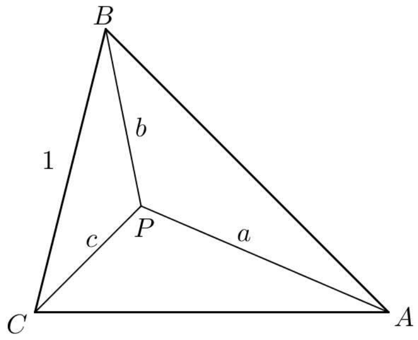


Fig. 9. Illustration of Problem 4

An elementary argumentation shows that equilateral triangle admits no division into three ideal triangles.

**Example 5.1.** Let $P$ be a point inside an equilateral triangle $ABC$. Then the measures of angles $\angle APC$, $\angle APB$ and $\angle BPC$ are greater than $\pi/3$. Denote the triangles $ACP$, $BCP$ and $ABP$ by $\Delta_1$, $\Delta_2$ and $\Delta_3$, respectively. Suppose that every $\Delta_i$ is an ideal triangle with ratio $y_i$, $i \in \{1, 2, 3\}$. Let $\alpha_i$, $\beta_i$, $\gamma_i$ be the angles of $\Delta_i$ with $\alpha_i < \beta_i < \gamma_i$. Clearly $P$ is the common vertex of $\gamma_1$, $\gamma_2$ and $\gamma_3$.

Observe that there are two angles $\alpha_i$, $\alpha_j$ having the same vertex $A$, $B$ or $C$. Suppose that $\angle ACP = \alpha_1$, $\angle CBP = \alpha_2$ and $\angle BAP = \alpha_3$. Then $|AP| \leq |CP| \leq |BP| \leq |AP|$, because the longer side is always across from the larger angle. Since the triangles $\Delta_i$ are ideal and isosceles, they must be equilateral. Thus $\gamma_1 + \gamma_2 + \gamma_3 = \pi \neq 2\pi$.

Without loss of generality we can assume that $\angle ACB = \alpha_1 + \alpha_2$. Then

$$y_1 = \frac{|AC|}{|CP|} = \frac{|BC|}{|CP|} = y_2,$$

so

$$|AP| = \frac{|CP|}{y_1} = \frac{|CP|}{y_2} = |BP|.$$

It follows that $\Delta_2$ is equilateral which contradicts the fact that $\gamma_2 > \pi/3$.

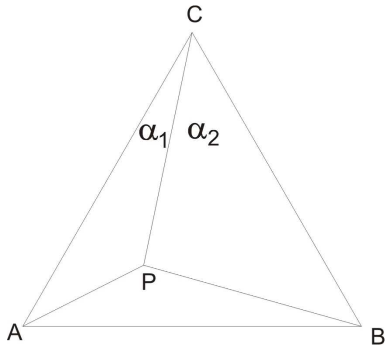


Fig. 10. Division of the equilateral triangle

**Example 5.2.** We know that there are at least two different isosceles triangles satisfying this property. Both of them are the unions of an equilateral and two ideal triangles with the largest angle $\gamma = 150^o$, as it shown in Figure 11.

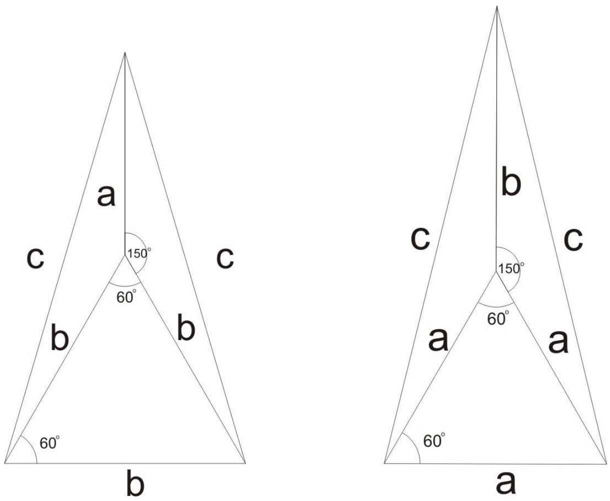


Fig. 11. Isosceles triangles divisible into three ideal triangles

We still do not know whether there exist other examples of triangles admitting the division into three ideal triangles. On the way of direct calculations we verified six general cases:

- $\frac{\phi}{a} = \frac{a}{b}$ with subcases
  - $c > a$ or
  - $a > c > b$ or
  - $b > c$ or

- $\frac{\phi}{b} = \frac{b}{a}$ with subcases
  - $c > b$ or
  - $b > c > a$ or
  - $a > c$

and we proved that the Kepler triangle does not possess this property.

As a curiosity let us notice that while verifying many cases we had to determine if it is possible that $a = \phi^{0.3}$, $b = \phi^{-0.4}$, $c = \phi^{0.4}$, $a = \phi^{0.6}$, $b = \phi^{0.2}$, $c = \phi^{0.4}$, $a = \phi^{-0.125}$, $b = \phi^{0.125}$, $c = \phi^{0.7}$ or $a = \phi^{0.375}$, $b = \phi^{-0.25}$, $c = \phi^{0.25}$ – only on the way of numerical verification, by using the Mathematica software and the Heron formula, we could exclude these cases. Let us also notice that the best approximation of this problem was given by the case of $a = \phi^{-0.75}$, $b = \phi^{-0.125}$, $c = \phi^{0.125}$, for which we obtained the covering of the Kepler triangle with the excess equal to 0.268664.

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